ALEXANDER DUALITY FOR MONOMIAL IDEALS ASSOCIATED WITH ISOTONE MAPS BETWEEN POSETS

JÜRGEN HERZOG, AYESHA ASLOOB QURESHI AND AKIHIRO SHIKAMA

Abstract. For a pair (P, Q) of finite posets the generators of the ideal $L(P, Q)$ correspond bijectively to the isotone maps from P to Q. In this note we determine all pairs (P, Q) for which the Alexander dual of $L(P, Q)$ coincides with $L(Q, P)$, up to a switch of the indices.

Introduction

In [5], Hibi and the first author introduced a class of monomial ideals which nowadays are called Hibi ideals. Given a finite poset P, the generators of Hibi ideals are squarefree monomials which correspond bijectively to the poset ideals of P. Later this class of ideals was generalized by Ene, Mohammadi and the first author in [2] by considering squarefree monomial ideals whose generators correspond to the chains of poset ideals of given length in P. The ideals generated by such monomials are called generalized Hibi ideals. In that paper, the Alexander dual of a generalized Hibi ideal is determined and is identified as a multichain ideal. The concept of generalized Hibi ideals and multichain ideals has been further generalized in [3] by Fløystad, Greve and the first author. To describe this class of ideals, let P and Q be finite posets. A map $\varphi : P \to Q$ is called *isotone* if it is order preserving. In other words, $\varphi : P \to Q$ is isotone if and only if $\varphi(p_1) \leq \varphi(p_2)$ for all $p_1, p_2 \in P$ with $p_1 \leq p_2$. The set of isotone maps $P \to Q$ is denoted by $\mathrm{Hom}(P, Q)$. Now let K be a field and S be the polynomial ring over K in the indeterminates x_{pq} with $p \in P$ and $q \in Q$. As in [3], we denote by $L(P, Q)$ the ideal generated by the monomials $u_\varphi = \prod_{p\in P} x_{p\varphi(p)}$ where $\varphi \in \mathrm{Hom}(P, Q)$. Let $[n]$ be the totally ordered poset with $1 < 2 < \cdots < n$. It is easily seen that a generalized Hibi ideal on P is of the form $L(P, [n])$ while a multichain ideal on Q is of the form $L([n], Q)$. In [3], the ideals $L([n], Q)$ and $L(P, [n])$ are called letterplace and co-letterplace ideals, respectively. The classical Hibi ideals can be identified with $L(P, [2])$.

According to Theorem 1.1 in [2], the Alexander dual $L(P, [n])^\vee$ of $L(P, [n])$ is equal to the ideal $L([n], P)^\tau$. Here, for any P and Q, $L(Q, P)^\tau$ is obtained from $L(Q, P)$ by switching the indices. In the other words,

$$L(Q, P)^\tau = (\prod_{q\in Q} x_{\psi(q)q} :\ \psi \in \mathrm{Hom}(Q, P)).$$

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An alternative proof of this fact is given [3, Proposition 1.2]. Since $(I^\vee)^\vee = I$ for any squarefree monomial ideal, one also has $L([n], P)^\vee = L(P, [n])^\tau$.

One would expect that $L(P,Q)^\vee = L(Q,P)^\tau$ in general. Unfortunately, this is not always the case as can be shown by simple examples. In this paper we determine all pairs (P,Q) of posets for which this duality holds. For this classification, we use [3, Lemma 1.1] which says that any isotone map $\varphi\colon P \to P$ of a finite poset P has a fixpoint, given that P has a unique minimal or maximal element.

1. Alexander duality for $L(P,Q)$

All posets considered in this paper are assumed to be finite. Recall that the direct sum of two posets P and Q on disjoint sets is the poset $P+Q$ on $P \cup Q$ such that $x \le y$ in $P+Q$ if either $x, y \in P$ and $x \le y$ in P or $x, y \in Q$ and $x \le y$ in Q. A poset is called *connected* if it is not a direct sum of two posets. Alternatively, P is connected if for any $a, b \in P$, there exists a finite sequence $a = a_1, a_2, \ldots, a_n = b$ in P such that a_i and a_{i+1} are comparable in P for $i = 1, \ldots, n-1$.

Let $\mathrm{Min}(L(P,Q))$ denotes the set of minimal prime ideals of $L(P,Q)$. By using [4, Corollary 1.5.5] it follows immediately that $L(P,Q)^\vee = L(Q,P)^\tau$ if and only if

$$\mathrm{Min}(L(P,Q)) = \{\mathfrak{p}_\psi \colon \psi \in \mathrm{Hom}(Q,P)\}, \tag{1}$$

where

$$\mathfrak{p}_\psi = (x_{\pi(q)q} \colon\ q \in Q). \tag{2}$$

In [3, Proposition 1.5] the following result is shown

Proposition 1.1. *Let P and Q be posets and assume that P has a unique maximal or minimal element. Then for any $\mathfrak{p} \in \mathrm{Min}(L(P,Q))$ with* $\mathrm{height}\,\mathfrak{p} \le |Q|$, *there exists $\psi \in \mathrm{Hom}(Q,P)$ such that $\mathfrak{p} = \mathfrak{p}_\psi$, and $\mathfrak{p}_\psi \ne \mathfrak{p}_{\psi'}$ for $\psi, \psi' \in \mathrm{Hom}(Q,P)$ with $\psi \ne \psi'$.*

As an immediate consequence of Proposition 1.1 one obtains

Corollary 1.2. *Let P and Q be posets and assume that P has a unique maximal or minimal element. Then*

(a) $\mathrm{height}\, L(P,Q) = |Q|$;
(b) $L(P,Q)^\vee = L(Q,P)^\tau$ *if and only if* $\mathrm{height}\,\mathfrak{p} = |Q|$ *for all* $\mathfrak{p} \in \mathrm{Min}(L(P,Q))$.

We first show

Proposition 1.3. *Let P and Q be posets such that $L(P,Q)^\vee = L(Q,P)^\tau$. Then P or Q is connected.*

Proof. Suppose that P and Q are both disconnected. Then there exists posets P_1, P_2 and Q_1, Q_2 such that $P = P_1 + P_2$ and $Q = Q_1 + Q_2$ with posets P_1, P_2 and Q_1, Q_2. Since $L(P,Q)^\vee = L(Q,P)^\tau$ it follows that $\mathrm{Min}(L(P,Q)) = \{\mathfrak{p}_\psi \colon\ \psi \in \mathrm{Hom}(Q,P)\}$. Let $p_1 \in P_1$ and $p_2 \in P_2$. Then the map

$$\psi(q) = \begin{cases} p_1, & \text{if } q \in Q_1, \\ p_2, & \text{if } q \in Q_2 \end{cases}$$

is isotone, and hence

$$\mathfrak{p}_\psi = (\{x_{p_1q}\colon\ q \in Q_1\} \cup \{x_{p_2q}\colon\ q \in Q_2\})$$

is a minimal prime ideal of $L(P,Q)$.

On the other hand, let $q_1 \in Q_1$ and $q_2 \in Q_2$ and let

$$\varphi(p) = \begin{cases} p_2, & \text{if } p \in Q_1, \\ p_1, & \text{if } p \in Q_2. \end{cases}$$

Then $\varphi\colon\ P \to Q$ is isotone and hence $u_\varphi = \prod_{p\in P_1} x_{pq_2} \prod_{p \in P_2} x_{pq_1}$ belongs to $L(P,Q)$, while $u_\varphi \notin \mathfrak{p}_\psi$, a contradiction. $\square$

In further discussion we may assume that P or Q is connected. In the next statement we will assume that P is connected.

We call P a *rooted* poset if for any two incomparable elements $p_1, p_2 \in P$ there is no element $p \in P$ such that $p > p_1, p_2$. Similarly we call P a *co-rooted* poset if for any two incomparable elements $p_1, p_2 \in P$ there is no element $p \in P$ such that $p < p_1, p_2$. Note that a poset which is rooted and co-rooted is a finite direct sum of totally ordered posets. Also, observe that if P is connected and rooted then P has a unique minimal element. Indeed, if P has two distinct minimal element, say $a, b \in P$, then by using the definition of connected poset, we obtain a sequence $a = a_1, a_2, \ldots, a_n = b$ in P such that a_i and a_{i+1} are comparbable, for all $i = 1, \ldots, n-1$. This sequence is not a chain because a and b are incomparable. Thus, there exist three distinct elements $a_{i-1} < a_i > a_{i+1}$ for some $i = 2\ldots, n-1$, which contradicts the definition of rooted poset. Similarly, if P is connected and co-rooted then P has a unique maximal element.

Theorem 1.4. *Let P and Q be finite posets, and assume that P is connected but not a chain.*

(a) *If P is rooted, then $L(P,Q)^\vee = L(Q,P)^\tau$ if and only if Q is rooted.*
(b) *If P is co-rooted, then $L(P,Q)^\vee = L(Q,P)^\tau$ if and only if Q is co-rooted.*
(c) *If P is neither rooted nor co-rooted, then $L(P,Q)^\vee = L(Q,P)^\tau$ if and only if Q is a direct sum of chains.*

Proof. (a) Assume that $L(P,Q)^\vee = L(Q,P)^\tau$ and that Q is not rooted. Then there exists $q_1, q_2, q_3 \in Q$ such that q_1 and q_2 are incomparable and $q_1, q_2 < q_3$. Let $p_1, p_2 \in P$ be a pair of incomparable elements. Since P is rooted and not a chain, there exists $p_3 \in P$ such that $p_3 < p_1, p_2$. We claim that

$$\mathfrak{p} = (\{x_{p_1q}\ :\ q \geq q_1\} \cup \{x_{p_2q}\ :\ q \geq q_2\} \cup \{x_{p_3q}\ :\ q \ngeq q_1 \text{ and } q \ngeq q_2\})$$

is a minimal prime ideal of $L(P,Q)$ with height $\mathfrak{p} > |Q|$. This will provide a contradiction to Corollary 1.2(b).

To prove our claim, we first show that $L(P,Q) \subset \mathfrak{p}$. Assume that there exists $\varphi \in \mathrm{Hom}(P,Q)$ such that $u_\varphi \notin \mathfrak{p}$. Then $\varphi(p_1) \ngeq q_1$ and $\varphi(p_2) \ngeq q_2$, and moreover, $\varphi(p_3) \geq q_1$ or $\varphi(p_3) \geq q_2$. We may assume that $\varphi(p_3) \geq q_1$. Then $q_1 \leq \varphi(p_3) \leq \varphi(p_1)$ contradicting the fact that $\varphi(p_1) \ngeq q_1$. Hence, $L(P,Q) \subset \mathfrak{p}$.

Now we show that $\mathfrak{p}$ is a minimal prime ideal of $L(P,Q)$. Due to Corollary 1.2, for all $q \in Q$, there exists $p \in P$ such that $x_{pq} \in \mathfrak{p}$. This implies that we can not

skip the variable x_{pq} from generators of $\mathfrak{p}$ if q appears only once as the second index. Assume now that $q \in Q$ appears twice as a second index among the generators of $\mathfrak{p}$. Then $q > q_1, q_2$ and $x_{p_1q}, x_{p_2q} \in \mathfrak{p}$. Now we show that we can not skip any of x_{p_1q} or x_{p_2q} from the set of generators of $\mathfrak{p}$.

Indeed, let $\psi : P \to Q$ given by

$$\psi(p) = \begin{cases} q, & \text{if } p \geq p_1, \\ q_1, & \text{otherwise.} \end{cases}$$

Note that ψ is an isotone map. In fact, let $p, p' \in P$ with $p \geq p'$. We have to show that $\psi(p) \geq \psi(p')$. This is obvious if $p, p' \geq p_1$ or $p, p' \ngeq p_1$. The only case which remains is that $p \geq p_1, p' \ngeq p_1$. But in this case we have $\varphi(p) = q > q_1 = \varphi(p')$.

Since ψ is an isotone map, it follows that $u_\psi \in \mathfrak{p}$. Since x_{p_1q} is the only generator of $\mathfrak{p}$ which divides u_ψ, this generator of $\mathfrak{p}$ can not be skipped. Similarly, one can show that x_{p_2q} can not be skipped as a generator of $\mathfrak{p}$. It shows that $\mathfrak{p}$ is indeed a minimal prime ideal of $L(P, Q)$.

Conversely, suppose that Q is a rooted poset and $L(P, Q)^\vee \neq L(Q, P)^\tau$. Then by using Corollary 1.2 (b), we see that there exists a minimal prime ideal $\mathfrak{p}$ of $L(P, Q)$ with $\operatorname{height} \mathfrak{p} > |Q|$. This implies that there exists an element $q \in Q$ such that $x_{p_1q}, x_{p_2q} \in \mathfrak{p}$ for some $p_1, p_2 \in P$ with $p_1 \neq p_2$. Since $\mathfrak{p}$ is a minimal prime ideal, neither x_{p_1q} nor x_{p_2q} can be skipped from the set of generators of $\mathfrak{p}$. It implies that there exist $\varphi_1, \varphi_2 \in \operatorname{Hom}(P, Q)$ such that x_{p_1q} is the only generator of $\mathfrak{p}$ which divides u_{φ_1} and x_{p_2q} is the only generator of $\mathfrak{p}$ which divides u_{φ_2}.

Suppose first that p_1 and p_2 are comparable. We may assume that $p_2 > p_1$. Then $\varphi_1(p_2) > q = \varphi_1(p_1)$, otherwise u_{φ_1} is divisible by both x_{p_1q} and x_{p_2q}. Similarly, $\varphi_2(p_1) < q = \varphi_2(p_2)$.

Let $\psi : P \to Q$ given by

$$\psi(p) = \begin{cases} \varphi_1(p), & \text{if } p \geq p_2, \\ \varphi_2(p), & \text{otherwise.} \end{cases}$$

We claim that ψ is an isotone map. To see this it suffices to show that $\psi(p) \geq \psi(p')$ for $p > p'$ and $p \geq p_2$, $p' \ngeq p_2$. Suppose that $p' < p_2$ then $\psi(p) = \varphi_1(p) > q > \varphi_2(p) = \psi(p')$. Suppose that $p' \nless p_2$, then p' and p_2 are incomparable. This case is not possible since $p > p_2$ and $p > p'$ and since P is rooted.

Following the construction of ψ, we see that $u_\psi \notin \mathfrak{p}$. This contradicts the fact that $L(P, Q) \subset \mathfrak{p}$.

Finally assume that p_1 and p_2 are incomparable. Since P is rooted, there exists a unique maximal element $p_3 \in P$ such that $p_3 < p_1, p_2$. Therefore, $\varphi_1(p_3), \varphi_2(p_3) \leq q$. Since Q is rooted, it follows that $\varphi_1(p_3), \varphi_2(p_3)$ are comparable. We may assume that $\varphi_1(p_3) \leq \varphi_2(p_3)$.

There exists a unique element p_4 with the property $p_3 < p_4 \leq p_1$, because P is rooted. Let $\psi : P \to Q$ given by

$$\psi(p) = \begin{cases} \varphi_2(p), & \text{if } p_4 \leq p, \\ \varphi_1(p), & \text{otherwise.} \end{cases}$$

We claim that ψ is an isotone map. To prove this it suffices to show that $\psi(p) > \psi(p')$ for $p > p'$ with $p \geq p_4$, $p' \not\geq p_4$. If $p' < p_4$ then note that $p' \leq p_3 < p_4$ because P is rooted. Then $\psi(p) = \varphi_2(p) \geq \varphi_2(p_4) \geq \varphi_2(p_3) \geq \varphi_1(p_3) \geq \varphi_1(p') = \psi(p')$. If p' and p_4 are incomparable then p and p' are also incomparable because P is rooted. It shows that ψ is an isotone map and $u_\psi \notin \mathfrak{p}$, a contradiction.

Statement (b) is proved in the same way.

(c) Let $L(P,Q)^\vee = L(Q,P)^\tau$ and assume that Q is not a direct sum of chains. Then there exists $q_1, q_2, q_3 \in Q$ such that q_1 and q_2 are incomparable and either $q_3 < q_1, q_2$ or $q_1, q_2 < q_3$. Assume that $q_1, q_2 < q_3$. Since P is neither rooted nor co-rooted we have p_1, p_2, p_3 such that p_1 and p_2 are incomparable and $p_3 < p_1, p_2$.Then by following the proof of (a) we obtain a minimal prime ideal of $L(P,Q)$ of height greater than $|Q|$, which is not possible. Similarly, one can show that it is not possible to have $q_1, q_2, q_3 \in Q$ such that q_1 and q_2 are incomparable and $q_3 < q_1, q_2$. It follows that Q is a direct sum of chains.

Conversely, assume that Q is the direct sum of the chains $Q_1, Q_2, \ldots, Q_n$. Then $L(P,Q)^\vee = (L(P,Q_1) + \cdots + L(P,Q_n))^\vee = L(P,Q_1)^\vee \cdots L(P,Q_n)^\vee$. By [3, Proposition 1.2], $L(P,Q_i) = L(Q_i,P)^\tau$. Therefore,

$$L(P,Q)^\vee = \prod_{i=i}^{n} L(Q_i,P)^\tau = (\prod_{i=i}^{n} L(Q_i,P))^\tau = L(Q,P)^\tau$$

□

As the final conclusion we obtain

Corollary 1.5. *The following conditions are equivalent:*

(a) $L(P,Q)^\vee = L(Q,P)^\tau$.

(b) *P is connected or Q is connected, and one of the following conditions is satisfied:*

  (i) *P and Q are rooted;*

  (ii) *P and Q are co-rooted;*

  (iii) *P is connected and Q is a sum of chains;*

  (iv) *Q is connected and P is a sum of chains;*

  (v) *P is a chain or Q is a chain.*

Proof. The result follows [2, Theorem 1.1], Proposition 1.3 and Theorem 1.4 observing that

$$L(P,Q)^\vee = L(Q,P)^\tau \iff L(Q,P)^\vee = L(P,Q)^\tau. \tag{3}$$

The statement (3) is a consequence of the fact that Alexander duality as well as the operator τ are involutary and commute with each other. Thus if $L(P,Q)^\vee = L(Q,P)^\tau$, then

$$(L(Q,P)^\vee)^\tau = (L(Q,P)^\vee)^\tau = (L(P,Q)^\tau)^\tau = L(P,Q),$$

which implies that $L(Q,P)^\vee = L(P,Q)^\tau$. This show " $\Rightarrow$ ". The other direction follows by symmetry. □

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Jürgen Herzog, Fachbereich Mathematik, Universität Duisburg-Essen, Campus Essen, 45117 Essen, Germany

E-mail address: `juergen.herzog@uni-essen.de`

Ayesha Asloob Qureshi, Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: `ayesqi@gmail.com`

Akihiro Shikama, Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: `a-shikama@cr.math.sci.osaka-u.ac.jp`